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Unit - 3

Symmetric Functions of the Roots

Symmetric functions of the roots of an equation are those in which all the roots are alike involved, so that function is unaltered in value when any two of the roots are interchanged.

Thus if α, β, γ be roots of a cubic, $\Sigma \alpha^2\beta^2$ represents the symmetric function

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$$

Again $\Sigma \alpha^2\beta$ represents

$$\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta$$

where all possible permutations of the roots two by two are taken and first root in each term is squared.

For a biquadratic whose roots are $\alpha, \beta, \gamma, \delta$,

$$\Sigma \alpha^2\beta^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2$$

EXAMPLES

① Find the value of $\Sigma \alpha^2\beta$ of the roots of cubic $x^3 + px^2 + qx + r = 0$

Sol. As

$$\alpha + \beta + \gamma = -p \quad \text{——} ①$$

$$\beta\gamma + \gamma\alpha + \alpha\beta = q \quad \text{——} ②$$

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Multiplying ① & ②,

$$(\alpha+\beta+\gamma)(\beta\gamma+\gamma\alpha+\alpha\beta) = -pq$$

$$\Rightarrow \alpha(\beta\gamma+\gamma\alpha+\alpha\beta)+\beta(\beta\gamma+\gamma\alpha+\alpha\beta) + \gamma(\beta\gamma+\gamma\alpha+\alpha\beta) = -pq$$

$$\Rightarrow (\alpha\beta\gamma+\alpha^2\gamma+\alpha^2\beta)+(\beta^2\gamma+\alpha\beta\gamma+\beta^2\alpha) + (\gamma^2\beta+\gamma^2\alpha+\alpha\beta\gamma) = -pq$$

$$\Rightarrow 3\alpha\beta\gamma+(\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\alpha+\gamma^2\beta) = -pq$$

$$\Rightarrow 3\alpha\beta\gamma+\Sigma\alpha^2\beta = -pq$$

$$\Rightarrow -3r+\Sigma\alpha^2\beta = -pq \quad \text{—} \left[\because \alpha\beta\gamma = -r\right]$$

$$\Rightarrow \boxed{\Sigma\alpha^2\beta = 3r-pq}$$ Answer

② Find for the same cubic $x^3+px^2+qx+r=0$ the value of $\alpha^2+\beta^2+\gamma^2$ & $\alpha^3+\beta^3+\gamma^3$.

Prof.

I $\quad \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$$= (-p)^2 - 2(q)$$

$$= p^2-2q$$ Answer

II Multiplying the values of $\Sigma\alpha$ & $\Sigma\alpha^2$, we get

$$(\Sigma\alpha)(\Sigma\alpha^2) = (-p)(p^2-2q)$$

$$\Rightarrow (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2) = -p^3+2pq$$

$$\Rightarrow \alpha(\alpha^2+\beta^2+\gamma^2) + \beta(\alpha^2+\beta^2+\gamma^2) + \gamma(\alpha^2+\beta^2+\gamma^2) = -p^3+2pq$$

$$\Rightarrow (\alpha^3+\alpha\beta^2+\alpha\gamma^2) + (\beta\alpha^2+\beta^3+\beta\gamma^2) + (\alpha^2\gamma+\beta^2\gamma+\gamma^3) = -p^3+2pq$$

$$\Rightarrow (\alpha^3+\beta^3+\gamma^3) + (\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\alpha+\alpha^2\beta+\gamma^2\beta) = -p^3+2pq$$

$$\Rightarrow \Sigma\alpha^3 + \Sigma\alpha^2\beta = -p^3+2pq$$

$$\Rightarrow \Sigma\alpha^3 \cancel{+} (3r-pq) = -p^3+2pq$$

$$\Rightarrow \boxed{\Sigma\alpha^3 = -p^3+3pq-3r}$$ Answer

(3) Find for same cubic the value of

$$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2.$$

Sol. As from eqn (2),

$$\beta\gamma + \gamma\alpha + \alpha\beta = q$$

Squaring on both sides, we get

$\Rightarrow (\beta\gamma)^2 + (\gamma\alpha)^2 + (\alpha\beta)^2 + 2(\beta\gamma)(\gamma\alpha)$
$+ 2(\gamma\alpha)(\alpha\beta) + 2(\alpha\beta)(\beta\gamma) = q^2$

$\Rightarrow (\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2) + 2\alpha\beta\gamma^2$
$+ 2\alpha^2\beta\gamma + 2\alpha\beta^2\gamma = q^2$

$\Rightarrow (\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2) + 2\alpha\beta\gamma(\gamma+\alpha+\beta) = q^2$

$\Rightarrow (\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2) + 2(-r)(-p) = q^2$

$\Rightarrow \boxed{\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2 = q^2 - 2pr}$ Answer

(4) Find for same cubic the value

of $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

Sol. Here, $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

$= [(\beta+\gamma)(\gamma+\alpha)](\alpha+\beta)$

$= (\beta\gamma + \alpha\beta + \gamma^2 + \alpha\gamma)(\alpha+\beta)$

$= (\alpha\beta\gamma + \alpha^2\beta + \alpha\gamma^2 + \alpha^2\gamma)$
$+ (\beta^2\gamma + \alpha\beta^2 + \gamma^2\beta + \alpha\beta\gamma)$

$= 2\alpha\beta\gamma + \Sigma\alpha^2\beta = 2(-r) + (3r - pq)$
$= r - pq$ Answer

(5) Find the value of symmetric functions

$\alpha^2\beta\gamma + \alpha^2\beta\gamma + \alpha^2\gamma\delta + \beta^2\alpha\gamma + \beta^2\alpha\delta + \beta^2\gamma\delta + \gamma^2\alpha\beta + \gamma^2\alpha\delta + \gamma^2\beta\delta + \delta^2\alpha\beta + \delta^2\alpha\gamma + \delta^2\beta\gamma$

of roots of biquadratic

$$x^4 + px^3 + qx^2 + rx + s = 0$$

Sol: As given,

$$x^4 + px^3 + qx^2 + rx + s = 0$$

Here, $\alpha + \beta + \gamma + \delta = -p$ — (1)

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$ — (2)

Multiplying (1) & (2), we get

$$\Sigma\alpha^2\beta\gamma + 4\alpha\beta\gamma\delta = pr$$

$$\Rightarrow \Sigma\alpha^2\beta\gamma + 4s = pr$$

$$\Rightarrow \Sigma\alpha^2\beta\gamma = pr - 4s \quad \text{Answer}$$

Exercise :-

(1) Find for some biquadratic the symmetric function

(i) $\sum \alpha^2$ Ans :- $p^2 - 2q$

(ii) $\sum \alpha^2 \beta^2$ Ans :- $q^2 - 2pr + 2s$

(iii) $\sum \alpha^3 \beta$ Ans :- $p^2 q - 2q^2 - pr + 4s$

(iv) $\sum \alpha^4$ Ans :- $p^4 - 4p^2 q + 2q^2 + 4pr - 4s$